Lect21-0405a Compactness

Heine-Borel Every open cover has a finite subcover

Other related compactness

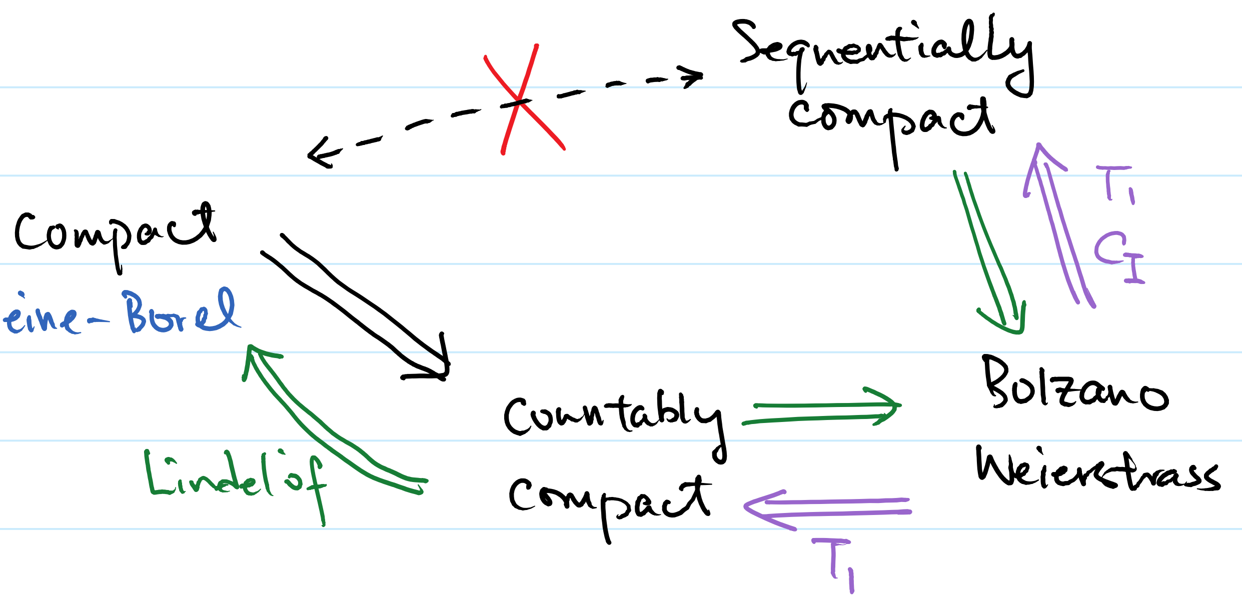
Countably Compact Every countable open cover has a finite subcover

Bolzano-Weierstrass Every infinite set has a cluster point in X.

If $A \subset X$ is infinite then $\exists\, x \in X$ s.t. $\forall\, U \in \mathcal{T}$ with $x \in U$, $U \cap A \setminus \{x\} \neq \emptyset$

Sequentially compact Every sequence has a convergent subsequence.

$\forall$ sequence (x_n) in X, $\exists$ subsequence (x_{n_k}) such that $x_{n_k} \longrightarrow x \in X$.



Sequentially Compact $\Longrightarrow$ Bolzano Weierstrass

Let $A \subset X$ be infinite

Create an infinite sequence $(a_n)_{n\in\mathbb{N}}$ in A

Get a convergent subsequence $a_{n_k} \longrightarrow x \in X$

Expect that $x \in A'$

Let $U \in \mathcal{J}$ with $x \in U$

By $a_{n_k} \longrightarrow x$, $\exists\, k_0 \in \mathbb{N}$ s.t. $\forall\, k \geq k_0$

$$a_{n_k} \in U$$

$$\therefore \quad a_{n_k} \in U \cap A$$

How & why $a_{n_k} \in U \cap A \setminus \{x\}$

Method. Pick a distinct sequence

$a_n \in A$, i.e., $a_m \neq a_n$

$\therefore$ The set $\{a_n : n \in \mathbb{N}\}$ is infinite

$\exists\, a_{n_k} \longrightarrow x \in X$,

If $x = a_{n_l}$ for some $l \in \mathbb{N}$

then remove a_{n_l}

We have a subsequence $(a_{n_k})_{k\in\mathbb{N}}$

such that

* $a_{n_k} \longrightarrow x$ as $k \to \infty$
* $a_{n_k} \neq a_{n_j}$ $\forall\, k, j \in \mathbb{N}$
* $a_{n_k} \neq x$ $\forall\, k \in \mathbb{N}$

Bolzano-Weierstrass $\xRightarrow[T_1]{C_I}$ Sequentially compact

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X

Consider $A=\{x_n : n\in\mathbb{N}\}$ Is it infinite?

If A is finite, $\exists$ constant subsequence and it converges

Assume that A is infinite, by Bolzano-Weierstrass, $\exists$ $x\in A'$, i.e. $\forall$ $U\in\mathcal{T}$ with $x\in U$, $\phi\neq U\cap A\setminus\{x\}$

As X is C_I, let $\mathcal{U}=\{U_k : k\in\mathbb{N}\}$ be a local base at x. Then $U_k\cap A\setminus\{x\}\neq\phi$

Qu. How to pick a subsequence in $U_k\cap A$ and make sure it converges?

* First, since $A=\{x_n : n\in\mathbb{N}\}$ is infinite, so is the set $A\setminus\{x\}$

  We may assume $x_n\neq x$ and $x_m\neq x_n$ $\forall$ m,n
* Pick $x_{n_1}\in U_1\cap A\setminus\{x\}$
* Consider $V_2=U_1\cap U_2\setminus\{x_1,x_2,\cdots,x_{n_1}\}\in\mathcal{T}$ because X is T_1.

  $\exists$ $x_{n_2}\in V_2\cap A\setminus\{x\}$
* Similarly, $x_{n_k}\in V_k\cap A\setminus\{x\}$ where $V_k=U_1\cap\cdots\cap U_k\setminus\{x_1,x_2,\cdots,x_{n_{k-1}}\}$

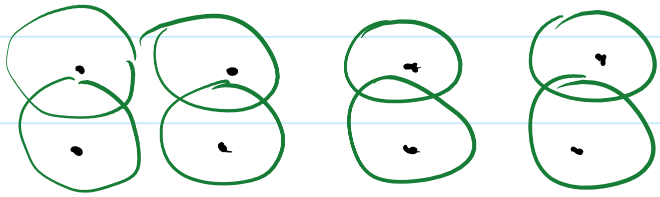
Countably Compact $\Longrightarrow$ Bolzano-Weierstrass

Qu. Think of an infinite set without cluster point

The obviously answer is $\mathbb{Z} \subset \mathbb{R}$ or $\mathbb{Z}^2 \subset \mathbb{R}^2$.

Qu. Find a countable cover for $\mathbb{R}^2$ which has no finite subcover

e.g. $\{B(x, \frac{1}{2}) : x \in \mathbb{Z}^2\} \cup \{\mathbb{R}^2 \setminus \mathbb{Z}^2\}$



Qu. Produce a proof for the contrapositive from this example.

Let $A \subset X$ be an infinite set and $A' = \emptyset$

Take a countable subset $B \subset A$, $B' = \emptyset$

1. B is discrete

   Let $b \in B$. As $B' = \emptyset$, $b \notin B'$

   $\therefore \exists\, U_b \in \mathcal{T}$ with $b \in U_b$ such that

   $U_b \cap B \setminus \{b\} = \emptyset$, i.e., $U_b \cap B = \{b\}$

2. $X \setminus B$ is open

   Let $x \in X \setminus B$. As $x \notin B'$

   $\exists\, U_x \in \mathcal{T}$ with $x \in U_x$, $U_x \cap B \setminus \{x\} = \emptyset$

   $\therefore U_x \setminus \{x\} \subset X \setminus B$

   $x \in U_x \subset X \setminus B$

Bolzano-Weierstrass $\xRightarrow{T_1}$ Countably Compact

Let $\{U_n : n \in \mathbb{N}\}$ be a countable open cover for X

How to get an infinite set?

Idea, if $U_1 \cup \cdots \cup U_n \not\supset X$ then $\exists\, x_n$ beyond!

But if $x_n \xrightarrow{\text{cluster}}$ at x, then many points can be covered by a single U_n

Take any $x_1 \in \bigcup_{n=1}^{\infty} U_n = X$

Then $x_1 \in U_{n_1}$ but $x_1 \notin U_1 \cup \cdots \cup U_{n_1 - 1}$

If $X \neq \bigcup_{n=1}^{n_1} U_n$ then we have

$$x_2 \in U_{n_2}, \quad x_2 \notin U_1 \cup \cdots \cup U_{n_1} \cup \cdots \cup U_{n_2 - 1}$$

Claim. The process must stop at finite step!

i.e. $X = \bigcup_{n=1}^{n_k} U_n$, finite subcover

Assume not, $\exists$ infinite $\{x_q : q \in \mathbb{N}\}$ where

$$x_q \in U_{n_q} \quad \text{but} \quad x_q \notin U_j \text{ for } j < n_q$$

We will show $A' = \phi$, i.e., $\forall\, x \in X$, $x \notin A'$

Since $X = \bigcup_{n=1}^{\infty} U_n$, $x \in U_m$ for some m.

Thus, we have $\quad n_N < m \leq n_{N+1}$

$$\underbrace{\begin{array}{cccccccccccc} U_1 & U_2 & \cdots & U_{n_1} & \cdots & U_{n_2} & \cdots & U_{n_N} & \cdots & U_m & \cdots & U_{n_{N+1}} \; \cdots \\ & & & \cup & & \cup & & \cup & & \cup & & \cup \\ & & & x_1 & & x_2 & & x_N & & x & & x_{N+1} \end{array}}_{\text{By construction, no } x_{N+1},\, x_{N+2},\, x_{N+3},\, \cdots}$$

X is T_1, $\therefore x \in V = U_m \setminus \{x_1, x_2, \cdots, x_N \text{ or } x_{N+1}\} \in \mathcal{T}$

But $V \cap A \setminus \{x\} = \phi$